Borel Quantisation
and
Nonlinear Quantum Mechanics.
A Review of Developments in the Series
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1   Introduction

In the first Symmetries in Science meeting in 1979 at Carbondale we presented preliminary results for a quantisation of the classical kinematic for non–relativistic systems which are localised and moving on a smooth manifold M. Our paper [1] 'On Global Properties of Quantum Systems' was published in the Proceedings of Symmetries in Science series. We developed subsequently (with Bernd Angermann) [2, 3] a quantisation method on smooth manifolds — the 'Quantum Borel Kinematics' (QBK); for a recent review see [4]. In 1992 a suitable time dependence was proposed (with Jerry Goldin) (see the review [5] and a more general 'Borel Quantisation' (BQ) which emerged from geometrical and topological considerations; it indicated a nonlinear extension of quantum mechanics. We participated in some of the later editions of Symmetries in Science series, often together with members of the 'Clausthal group', e.g. Vlado Dobrev, Jerry Goldin, Wieland Groth, Jörg Hennig, Wolfgang Lücke, Hans-Jürgen Mann, Peter Nattermann, Wolfgang Scherer, Christoph Schulte, Pavel Šťovíček and Reidun Twarock. The results, different aspects and applications of Borel Quantisation can be found in the volumes of 'Symmetries in Science'.

Our interest in quantum mechanics on manifolds was connected with the

following situation: During 1970–1980 some of our colleagues in quantum theory and in particle physics thought that Lie groups and their representations are a major key to model and to understand particle physics. In this context we worked e.g. on spectrum generating algebras and on embeddings of physical Lie algebras. Based on Mackey's theory of induced representations we wrote a paper [6] on a quantisation of particles moving on homogeneous G–spaces. We realised that the geometry of the G–space does not contain 'enough' information for a time evolution on G. Furthermore, we failed to generalise Mackey's method to physical important non–homogeneous spaces. Group theory was obviously a very successful model, but it was too 'rigid': If one chooses the group and its representation, the complete mathematical framework is already given; there does not appear the flexibility which one wants for a description of physical systems. Hence those mathematical formalisms which are 'close' to group theory and which are in addition more 'flexible' became interesting. Among such formalisms are: nonlinear and non–integrable representations of Lie algebras and their deformations in the sense of Gerstenhaber. A further promising field for a geometric modelling are differential geometrical and algebraic notions on M. Here one views physical laws e.g. as relation between geometrical or algebraic objects living on M. Following the pioneering papers of George Mackey [7] and Irving Segal [8], we found a path to understand quantisations of a system on a topologically nontrivial configuration space and how the quantised system 'feels' the topology. This leads to Quantum Borel Kinematics characterised by topological quantum numbers and one additional quantum number D. This D is connected with the structure of the infinite–dimensional Lie algebra spanned by quantised kinematical operators.

In our kinematical design a physical interpretation of D was obscure. It became more transparent in 1988 when Jerry Goldin and one of the authors (HDD) realized that two approaches are equivalent: the quantised Borel kinematics on Euclidean configuration spaces and the representations of non–relativistic current algebras on multiparticle configuration spaces for indistinguishable objects found by Goldin and co-workers [9]. The quantum number D appears in front of an additional term in the generalised momentum operator as well as in the momentum current. Jerry Goldin and HDD introduced, based on this observation, a generic time dependence for pure states and derived a family of nonlinear Schrödinger equations [10] — DG equations — with nonlinear term proportional to D. Special generalisations to mixed states (von Neumann equations) are known [11]. A direct connection of the DG family to certain nonlinear gauge transformations [12, 13] and an interpretation of D through nonlinear transformation [14] was elaborated.

Some of these developments are reviewed and commented in this contribution.

2   Borel Kinematic

2.1   Classical Case

We start with a classical model for the kinematic of a system (particle) localised and moving on a smooth manifold M. The kinematic is characterised by the following set of observables:

Generalised positions
$$f \in C^\infty(M, R)$$

realised by real functions on M, and
generalised momenta
$$X \in Vect(M)$$

realised by smooth vectorfields on M.

We define the tuple

$$S(M) = (C^\infty(M, R), Vect(M))$$

as the generic KINEMATIC on M (or covariance algebra of M). $S(M)$ has the following properties:

1. $Vect(M)$ is a (∞–dimensional) Lie algebra of a subgroup of the diffeomorphism group $\mathrm{DIFF}(M)$ of M;

2. $C^\infty(M, R)$ can be viewed as an (∞–dimensional) Abelian Lie algebra;

3. f and X defined on M form a semidirect sum

$$S(M) = C^\infty(M, R) \oplus_s Vect(M).$$

For physical reasons (see section 2.3) we restrict $Vect(M)$ to the subset of complete vectorfields $Vect_0(M)$; this subset spans a partial Lie algebra) (the Lie bracket of two complete vector fields may not be complete) and the corresponding kinematic $S_0(M)$ has partial Lie algebra structure.

2.2 Quantisation of $S_0(M)$

To quantise the classical object $S_0(M)$ we construct a map from $S_0(M)$ into the set of essentially selfadjoint operators on a common dense domain in a separable Hilbert space H

$$\mathbb{Q} = (\mathsf{Q}, \mathsf{P}) : \quad S_0(M) \longrightarrow SA(H)$$

sending

$$f \longrightarrow \mathsf{Q}(f), \quad X \longrightarrow \mathsf{P}(X)$$

We realise H as $L^2(M, \mathbb{C}, d\nu)$, i.e. the space of square integrable complex functions over M; $d\nu$ is a standard measure on M. We assume furthermore that there is no internal degree of freedom like spin. The map can be viewed as a representation of an infinite dimensional (partial) Lie algebra.

The following properties I.-III. are assumed for $\mathbb{Q}$ to be a *quantisation map*:

I. In $L^2(M, \mathbb{C}, d\nu)$ operators $\mathsf{Q}(f)$ act as multiplication operators by f, i.e.

$$\mathsf{Q}(f)\Psi = f\Psi.$$

II. The Lie algebra structure of $S_0(M)$ survives, i.e. $\mathbb{Q}$ is a partial Lie algebra homomorphism.

III. $\mathsf{P}(X)$ is a local operator, i.e. $supp\ (\mathsf{P}(X)\Psi) \subset supp\ \Psi$.

These assumptions have the following background:

Ad **I.** Consider a set of localisation regions $B \subset M$; choose for this set the Borel field $\mathfrak{B}(M)$ over M and define the quantisation map:

$$\mathbb{Q} : B \in \mathfrak{B}(M) \longrightarrow E(B) \in SA(H).$$

The states of the system are given by normed positive trace–class operators W. The expectation value of a measurement of $E(B)$ in a state W is

$$Tr(WE(B)) = \mu_W(B) \tag{1}$$

($Tr(.)$ denotes trace) and contains information on the probability of localisation of the system in state W in the region B. Using properties of position measurements we assume that the r.h.s. of (1) is (elementary) spectral measure on $\mathfrak{B}(M)$. With our realisation of the spectral theorem we find property **I.**

Ad **II.** The algebraic structure of $S_0(M)$ reflects that the classical system is localised and moving on M. Also the quantum system lives on M. Hence this algebraic structure should 'survive' under the quantisation map. In this sense quantisations are based "on an algebra" [15].

Our later analysis shows that there are different possibilities for such maps which are related to unitarily inequivalent quantisations. Therefore the information encoded in the classical system is not sufficient to characterise its quantised form; one needs additional information — so called 'quantum information'.

Ad **III.** For $M = R^n$ we know that the momentum operator acts in H as a selfadjoint differential operator. It is plausible to expect also that $\mathsf{P}(X)$

acts locally on complex functions over M as differential operator. Hence in our design we have to define differential operators (of finite order) on functions in $L^2(M, \mathbb{C}, d\nu)$. For this we sketch two notions (**A, B**) and their relation (**C**):

A. To define derivatives of complex functions over M one needs a differentiable structure DS on the point set $M \times \mathbb{C}$. The restrictions of DS give the differentiable structure $DS(M)$ of M (smooth manifold)

$$DS(M \times \mathbb{C})|_M = DS(M)$$

and the restriction to $\mathbb{C}$ yields the standard differentiable structure of $\mathbb{C}$,

$$DS(M \times \mathbb{C})|_{\mathbb{C}} = DS(\mathbb{C})$$

Geometrical objects with these properties are complex line bundles on M with hermitian connection, $\mathfrak{L}(M \times \mathbb{C}, pr, \mathbb{C}, \langle . \rangle, \nabla)$. Some sections of $\mathfrak{L}$ are square integrable. The Hilbert space $L^2(M, \mathbb{C}, d\nu)$ can be viewed as the space of square–integrable sections of $\mathfrak{L}$.

The structure of the set $\{\mathfrak{L}\}$ of such bundles is known. Denoting the curvature of the connection ∇ in $\mathfrak{L}$ by R, one can construct such $\mathfrak{L}$ if and only if

$$\frac{1}{2\pi i} \int_S R \in \mathbb{Z}$$

for all closed 2-surfaces S in M. In terms of cohomology the de Rham class of $\frac{1}{2\pi i}R$ has to be integral,

$$\left[\frac{1}{2\pi i}R\right] \in H^2(M, \mathbb{Z}), \tag{2}$$

i.e. there is a strong bundle isomorphism between two complex line bundles, if and only if their Chern classes in $H^2(M, \mathbb{Z})$ coincide.

For each of these inequivalent classes there is a set of inequivalent connections labelled by

$$H^1(M, U(1)) = \pi_1^*(M),$$

i.e. by elements of the character group of the fundamental group of M.

These algebraic invariants classify the line bundles together with their differentiable structures and covariant derivatives ∇, we are looking for. We have no result whether the introduction of differentiable structures to our model via line bundles is unique. For internal degrees of freedom complex vector bundles can be used [16, 17].

B. For physical reasons we want to avoid nonlocal effects, i.e. we quantise the kinematic by local operators $\mathsf{Q}(f)$ and $\mathsf{P}(X)$. The position operators are local by construction; the locality of $\mathsf{P}(X)$ is equivalent to condition **III.** Note that we assume locality only for $\mathbb{Q}S_0(M)$; other operators representing observables could be nonlocal.

C. The locality condition is linked with differential operators defined via differentiable structures [18]: if there is a differentiable structure $DS(M \times \mathbb{C})$,

then the locality of $\mathsf{P}(X)$ implies that $\mathsf{P}(X)$ is a differential operator of finite order with respect to $DS(M \times \mathbb{C})$.

The arguments in section 2.2 are related to our generalisation of Mackey's imprimitivity theorem on homogeneous spaces. Our review [4] and references therein have utilised such a generalisation.

2.3 A Classification Theorem for Quantisation Maps

With the assumptions **I. − III.** we derived a classification theorem for the quantisation maps

Theorem

The set $\{\mathbb{Q}\}$ of unitarily inequivalent quantisation maps

$$\mathbb{Q}^{(.)}: \quad S_0(M) \longrightarrow SA(H)$$

on a Hilbert space H which is realised with square–integrable sections of a hermitian line bundle, $\mathbb{Q}(S_0(M))$, is labelled by the triple

$$(J, \alpha, D) \; \in \; H^2(M, \mathbb{Z}) \times \pi_1^*(M) \times R.$$

Here $H^2(M, \mathbb{Z})$ labels the set of closed two–forms J which satisfy the integrality condition (2), $\pi_1^*(M)$ denotes the character group of the fundamental group of M. For a fixed J we have classification by (α, D).

Explicit form of the map $\mathbb{Q}^{(\alpha,D)} = (\mathsf{Q}^{(\alpha,D)}, \mathsf{P}^{(\alpha,D)})$ can be found in [3, 4, 21]. Details for the case $M = R^3$ are given in section 2.4.

The operators $\mathsf{Q}^{(J,\alpha,D)}(f)$ and $\mathsf{P}^{(J,\alpha,D)}(X)$ are essentially selfadjoint (we used complete vector fields) on a common invariant domain; the representation $\mathbb{Q}^{(.)}(S_0(M))$ is irreducible; α is a topological quantum number; D is independent of the topology and is related to the algebraic structure of $S_0(M)$. Hence there are inequivalent quantisations for systems on topologically trivial manifolds.

REMARKS

1. Quantum Borel kinematics are based on a classical configuration space in contrast to geometric (pre–)quantisation [19] which works on a symplectic space or more specially on the phase space. In both methods the topological quantum numbers play (with different motivations) an essential role. However, the quantum number D appears only in quantum Borel kinematics. This gap was closed recently: Jörg Hennig and Peter Nattermann showed in [20, 21] that geometric quantisation of the kinematic corresponds to our approach, if one uses $(\frac{1}{2} - i\gamma)$–density instead of a $\frac{1}{2}$–density; the imaginary part of the density is proportional to the quantum number D.

2. We sketch in paragraphs A, B, C formulations of quantum Borel kinematic for more general situations:

A. If the system has internal degrees of freedom like spin, the Hilbert space is spanned by vector–valued functions. There are two types of quantum maps: type 0 in which different vector components are not mixed; this type is described in [16]. In type 1 a mixing is allowed; Michael Drees gave some preliminary results [17]. A discussion of the quantum Borel kinematic for a spinning particle can be found in [22, 23].

B. If there exists an external field or potential on M, i.e. a closed 2-form B, an additional term in the commutator of the momentum observables appears. Hence a quantisation map based on M exists only if B fulfils the integrality (admissibility) condition (2). We refer to [4, 21]. However, such external potentials were already included in an indirect way in our earlier discussion, since the quantum number J is responsible for the existence of both the line bundles and the admissible closed external one–forms.

C. The physical background of D cannot be completely clarified in a kinematical framework. One can calculate how expectation values of momenta depend on D. But this dependence can be analysed only if it is explicitly known. We discuss this in section 3.

2.4 Applications of the Classification Theorem

The theorem shows that for topologically trivial as well as non–trivial manifolds 'different quantisations' exist. This means that the probabilities of certain observables measured in certain states depend on (J, α, D). One and the same classical system yields — after the quantisation maps — a set of different quantum systems. Additional information — the already mentioned 'quantum information' — is necessary to choose or to determine (J, α, D). The source of this information can be, e.g., first principles or experimental results.

As the first example of inequivalent quantisations we consider a topologically trivial manifold $M = R^3$ with vector fields $X(\vec{g}) = \vec{g}(\vec{x}).\vec{\nabla}$ and with quantisation map $\mathbb{Q}^{(D)}(S_0(R^3))$,

$$\mathsf{Q}(f) = f$$

$$\mathsf{P}^{(D)}(X) = -i\hbar\vec{g}.\vec{\nabla} + (-i\frac{\hbar}{2} + D)\ div\ \vec{g} \tag{3}$$

Different D yield unitarily inequivalent representations and hence different quantum systems.

The representations of $S_0(R^n)$ for multiparticle configuration spaces for N indistinguishable objects can be viewed also as representations of non–relativistic inhomogeneous current algebras. Jerry Goldin and co-workers [9] constructed such representations; they derived the above result (3) and found independently the quantum number D.

Table 1: Examples of elementary quantum Borel kinematics [4].

Quantum system	M	$\pi_1(M)$	$H_1(M, Z)$	$H^2(M, Z)$	Topological quantum numbers
Spinless particle in R^3	R^3	$\{e\}$	0	0	—
Aharonov–Bohm configuration	$R^3 \backslash R$	Z	Z	0	$\vartheta \in [0, 1)$
Dirac's monopole	$R^3 \backslash O = R_+ \times S^2$	$\{e\}$	0	Z	$n \in Z$
2 distinguishable particles in R^3	$R^3 \times R_+ \times S^2$	$\{e\}$	0	Z	$n \in Z$
2 indistinguishable particles in R^3	$R^3 \times R_+ \times RP^2$	S_2	Z_2	Z_2	$m \in Z_2$
Rigid body	$R^3 \times SO(3)$	Z_2	Z_2	Z_2	$m \in Z_2$
Symmetric top	S_2	$\{e\}$	0	Z	$n \in Z$
Rotator with fixed axis	S^1	Z	Z	0	$\vartheta \in [0, 1)$
Particle on orientable surface of genus p	K_p	$\pi_1(K_p)$	Z^{2p}	Z	$n \in Z,$ $\vartheta_1 \ldots \vartheta_{2p} \in [0, 1)$

Now let us mention some systems on topologically non–trivial smooth manifolds and their different quantisations (see also Table 1):

The physics of N indistinguishable particles — anyons — and distinguishable particles on a 2–dimensional Euclidean space (in the framework of current algebra) was discussed by Jerry Goldin and co-workers. A review (in Borel quantisation) of indistinguishable and distinguishable particles on 2–dimensional manifolds can be found in [24]. Parastatistics appears for N indistinguishable particles on manifolds with dimension > 3 [24]. Aharonov–Bohm situations were discussed as topological effects in [25]. The quantum maps for systems on non–orientable 2–dimensional manifolds (Möbius strip and Klein bottle) were treated in [26]. For quantisations on the trefoil knot see [27]. A recent review of many aspects in quantum Borel kinematics can be found in [4] together with some further examples of configuration spaces with nontrivial topology.

3 Borel Dynamics

3.1 Difficulties with $\mathbb{Q}^{(\cdot)}(S_0(M))$

Quantum Borel Kinematic holds for any fixed time t, it considers a 'frozen' system and it carries no direct information on a t–dependence. Hence a principle is needed to construct related dynamical equations. As explained before one hopes that this could be a key for a physical interpretation of the quantum number D.

As a plausible model for an evolution of pure states we choose a dynamical group $\{\mathfrak{D}_{t,g}; t \in R\}$ with a linear operator $\mathbb{D}_g$ as generator acting on H; g denotes a Riemannian structure on M. In the Heisenberg picture we relate the quantised momentum $\mathsf{P}^{(\cdot)}(X)$ to $\mathbb{D}_g$, i.e. we assume the existence of a map

$$C^\infty(M, R) \to Vect_0(M)$$

such that

$$[\mathbb{D}_g, \mathsf{Q}(f)] = -i\mathsf{P}^{(\cdot)}(X_f) \quad \text{for all} \quad f \in C^\infty(M, R).$$

In analogy to Hamiltonian mechanics in phase space we specialise this map with

$$X_f = grad_g \ f.$$

To analyse this ansatz consider the ∇–lift of the Laplace Beltrami operator Δ_g on M (with metric g) is a candidate for $\mathbb{D}$. Hence we write (with some operator $\mathbb{K} = \mathsf{Q}(V)$)

$$\mathbb{D}_g = -\frac{1}{2}\Delta_g^\nabla + \mathbb{K}.$$

The commutator between $\mathbb{D}_g$ and f has with (3) for all $f \in C^\infty(M, R)$ the form

$$[\mathbb{D}_g, f] = i\mathsf{P}(grad_g \ f) - iD\Delta_g f.$$

A comparison with the previous result yields $D = 0$. Thus our ansatz for a dynamical group (in the Heisenberg picture) fails; it leads to the trivial result $D = 0$ [28]. This failure is partly connected with the implicit assumption that evolutions of wave functions are linear.

3.2 Nonlinear evolutions from $\mathbb{Q}^{(D)}(S_0(R^3))$

An alternative method is to start with the assumption that the positional probability is conserved. Consider again a system in $M = R^3$. We assumed [5]

$$\frac{\partial}{\partial t} \int_{R^3} \varrho(x, t) d^3 x = 0, \qquad \varrho(x, t) = \overline{\Psi}(x, t)\Psi(x, t).$$

This implies indirectly that a pure state remains a pure state. With a suitable behaviour of Ψ at infinity we are allowed to apply the Gauss theorem and find

$$\frac{\partial}{\partial t}\varrho(x, t) = -\nabla \vec{j}(x, t) \tag{4}$$

with a vector field density $\vec{j}$ depending on the wave function Ψ.

How to construct $\vec{j}$ in our model? Consider the above equation as an operator equation in the 1–particle sector $\mathfrak{F}_1$ of the Fock space generated from a cyclic vacuum $|0\rangle$. The generic operators $\mathsf{Q}(f)$, $\mathsf{P}^{(D)}(X)$ correspond to operator–valued densities ϱ, $\vec{j}^D$ in $\mathfrak{F}_1$:

$$\mathsf{Q}(f) = \int f(x)\varrho(x,t)d^3x$$

$$\mathsf{P}^{(D)}(X) = \int \vec{g}(x)\vec{j}^D(x,t)d^3x$$

We have already $\varrho(x,t) = \overline{\Psi}\Psi$. We get for $\vec{j}^D(x,t)$ from (3)

$$\vec{j}^D(x,t) = \vec{j}^0(x,t) - D\nabla\varrho(x,t), \quad \vec{j}^0(x,t) = \frac{\hbar}{2mi}(\overline{\Psi}\cdot\nabla\Psi - \overline{\nabla\Psi}\cdot\Psi)$$

and with (4)

$$\frac{\partial}{\partial t}\varrho(x,t) = -\nabla\vec{j}^0(x,t) + D\nabla\varrho(x,t).$$

This is a Fokker–Planck type equation. For $D = 0$ we have the quantum mechanical continuity equation with the usual quantum mechanical current. The term proportional to the quantum number D is a quantum mechanical diffusion current which is characteristic for the model.

Any ansatz for a time dependence of $\Psi(x,t)$ has to respect this Fokker Planck equation. We use this fact to construct an evolution of first order in ∂_t, with the usual linear terms and with an additional term F depending e.g. on the wave function Ψ:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = (-\frac{\hbar^2}{2m}\Delta + V(x) + F[\Psi])\Psi$$

Inserting this ansatz into the Fokker–Planck equation, a non linear Schrödinger equation with a complex nonlinear term is obtained,

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = (-\frac{\hbar^2}{2m}\Delta + V(x) + iIm\ F[\Psi] + Re\ F[\Psi])\Psi,$$

where

$$Im\ F[\Psi] = \hbar\frac{D}{2}\frac{\Delta\varrho}{\varrho}$$

is enforced through $\mathbb{Q}^{(D)}$, and $Re\ F[\Psi]$ is independent of D (arbitrary).

We see that the imaginary part of F is fixed by the quantisation method, but there is no information on the real part. We assume for Re F a function of the wave function which is of the same type as ImF, i.e.
  • complex homogeneous of order zero
  • rational with derivatives not higher than second order in the numerator
  • Euclidean invariant.

With these assumptions for *Re F* Doebner and Goldin obtained a family of singular nonlinear Schrödinger equations (DG equations) for a particle with mass m, potential V parametrised by $\hbar$, D, $c_1,...,c_5$:

$$i\hbar\frac{d}{dt}\Psi = \left[-\frac{\hbar^2}{2m}\Delta + V(\vec{x}) + \frac{1}{2}\hbar D\frac{\Delta\varrho}{\varrho} + \hbar D'\sum_{i=1}^{5}c_i R_i\left[\Psi\right]\right]\Psi$$

$$R_1\left[\Psi\right] = \frac{m}{\hbar}\frac{\nabla\vec{j}^0}{\varrho}, \quad R_2\left[\Psi\right] = \frac{\Delta\varrho}{\varrho}, \quad R_3\left[\Psi\right] = \frac{m^2}{\hbar^2}\frac{(\nabla\vec{j}^0)^2}{\varrho^2}$$

$$R_4\left[\Psi\right] = \frac{m^2}{\hbar^2}\frac{(\nabla\vec{j}^0)^2}{\varrho^2}, \quad R_5\left[\Psi\right] = \frac{(\nabla\varrho^2)}{\varrho^2}.$$

The choice of the real nonlinearity corresponds to the 'gauge generalisation' used in another derivation of the DG equations (see sections 4.1, 4.2).

Independently of the known fact (see e.g. the review [5]) that the usual framework of quantum mechanics does not allow fundamental nonlinear evolutions for pure states, one may argue that a small nonlinearity (small D) can be treated approximately with the usual methods. With this precaution some results for atomic spectra were derived; for the hydrogen atom present precession experiments show no difference to the linear theory. This leads to an upper bound for D [29]

$$D > 10^{-7}\frac{\hbar}{m}$$

There are discussions on DG type nonlinearities in quantum optics ('nonlinear photons') [29].

For an exact calculation of observable effects a new framework of quantum mechanics is necessary. There are indications how to formulate general requirements (e.g. [30]), but there is by no means a complete and mathematically acceptable theory which incorporates fundamental nonlinearities.

The mathematical structures and properties of DG equations and the mentioned precaution for their physical applications are partly known; we quote (the following list is incomplete):

Cauchy problem [31]; Lie symmetries [32]; solutions for stationary states [10], time dependent solutions for certain coefficients [33]; generalisations for: arbitrary smooth M [21], mixed states [11]; other methods for a derivation via: nonlinear gauge transformations [13], generalised Ehrenfest relations [11], stochastic processes [34]. Further applications are known for anti–particles [35] and the dynamics for D–branes.

4 Borel Kinematic and Nonlinear Structures

Our quantum number D yields a family of nonlinear evolution equations for pure states with a nonlinearity proportional to D. This indicates a hidden nonlinear structure in $\mathbb{Q}^{(0)}(S_0(R^3))$. We assume in the following $M = R^3$.

4.1 Nonlinear Gauge Transformations

In connection with the properties of stationary solutions of the DG family a group $\mathcal{G}$ of nonlinear gauge transformations was introduced [12]. They are invertible transformations $\mathbb{N}$

$$\mathbb{N} : \ \Psi \in H \longrightarrow \mathbb{N}\Psi = \mathbb{N}[\Psi] \in H \tag{5}$$

with $\mathbb{N}$ depending on $\Psi(x,t)$, x, t. They are restricted by the assumption that the positional probability density is invariant:

$$\overline{\mathbb{N}[\Psi]}\mathbb{N}[\Psi] = \overline{\Psi}\Psi.$$

The reason for this restriction is the following: $\mathbb{N}$ should transform a given system, i.e. a given Ψ, to a 'physically equivalent' one: "equivalent" means that the results of measurements on both systems are the same.

Behind the notion of 'physical equivalence' is the 'principle' that the positional density $\varrho(x,t)$ for all x and t determines the outcomes of the measurements of all observables [36]. Such $\mathbb{N}$ build a nonlinear gauge group $\mathcal{G}$.

Now applying $\mathbb{N}$ to the usual (linear) Schrödinger equation, i.e. to a system with linear evolution operator

$$\mathbb{D}_S = i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta - V(x), \quad \mathbb{D}_S\Psi = 0,$$

a (family of) nonlinear Schrödinger equations is obtained,

$$\mathbb{D}_S \circ \mathbb{N}[\Psi] = 0.$$

The result is a subfamily in the DG family.

By construction it describes the physics of a system which is equivalent to the linear system. A generic procedure (gauge generalisation) to construct 'new' systems is the gauge generalisation. This is a generic procedure for families of partial differential equations depending on coefficients which are related to each other; the breaking of this relation is our model for a "gauge generalisation" [12, 13]. This gauge generalisation leads to the DG family; some of its members are inequivalent, they represent systems with new physical properties.

Sections 3.2 and 4.1 show that the DG equations can be derived from two different structures:

- From a geometric structure via a representation of an inhomogeneous diffeomorphism group acting on the configuration space and with a t–dependence from a conservation of positional probability density.
- From a nonlinear structure via nonlinear transformations of positional probability densities between physically equivalent systems applied to a t–dependence of the linear system and gauge generalisation.

4.2 Nonlinear Tangent Map

Another method to describe a hidden nonlinear structure of $\mathbb{Q}^{(0)}(S_0(R^3))$ more directly was recently presented [14].

Let $\mathbb{N} \in \mathcal{N}$ as in (5) be a group of nonlinear transformations in H. To introduce convenient transformation properties for operators we consider physically interesting ones which are (often) essentially selfadjoint. They can be viewed as generators $i\mathbb{A}$ of a one parameter group U_ε of unitary transformations

$$U_t = \exp it\mathbb{A}.$$

Take a path $\{U_t\Psi, \ t \in [-\varepsilon, \varepsilon]\}$ in H. Then

$$\frac{d}{dt}(U_t\Psi)|_{t=0} = \frac{d}{dt}(\Psi + it\mathbb{A}\Psi)|_{t=0} = i\mathbb{A}\Psi.$$

Hence $i\mathbb{A}\Psi$ appears as a tangent map T of U_ε. Take now a transformed path $\mathbb{N}(U_\varepsilon)$ and define the transformed generator $i\mathbb{A}_\mathbb{N}$ by the tangent map $T(\mathbb{N})$ of $\mathbb{N}(U_\varepsilon)$:

$$\frac{d}{dt}\mathbb{N}(U_\varepsilon\Psi)|_{t=0} = i\mathbb{A}_\mathbb{N}\Psi.$$

Hence we have the $\mathbb{N}$-tangent map

$$T(\mathbb{N}): \quad \mathbb{A} \to \mathbb{A}_\mathbb{N}.$$

For linear $\mathbb{N}$ we get the usual result. For nonlinear $\mathbb{N}$ the resulting operator $\mathbb{A}_\mathbb{N}$ is in general nonlinear. The $\mathbb{N}$-tangent map is a Lie algebra isomorphism. One can extend the method formally to non essentially selfadjoint operators.

We apply now the $\mathbb{N}$-tangent map to quantise kinematical observables. Similarly as in section 4.1 we restrict $\mathbb{N}$ such that the $\mathbb{N}$-tangent mapped elements in $\mathbb{Q}^{(D)}(S_0(R^3))$ are again linear and of order 0 or 1, i.e.

1. $\mathsf{Q}(f)$ is a linear multiplication operator, i.e. f
2. $\mathsf{P}^{(D)}(X)$ is a linear differential operator of order 1.

Condition 1. is fulfilled by construction; condition 2. is equivalent to the relation

$$\mathsf{P}_\mathbb{N}^{(0)}(\vec{g} \cdot \nabla) = \vec{g}_1 \cdot \nabla + g_0$$

with $\vec{g}_1(x)$, $g_0(x)$ depending on $\vec{g}(x)$. The last condition implies [14] a set $\mathbb{N}$ of non linear transformations. We write this (formally) in polar decomposition

$$\Psi = R\exp iS, \qquad \mathbb{N}[\Psi] = \mathbb{N}[R, S] = R_\mathbb{N}(R, S)\exp iS_\mathbb{N}(R, S)$$

The form of $\mathbb{N}$ is

$$R_\mathbb{N}(R, S) = R^{\kappa+1}r(S)$$
$$S_\mathbb{N}(R, S) = \gamma \ln R + t(S) + 1$$

with $\kappa, \gamma \in R$, and real functions $r(S)$ and $t(S)$. The functions $\vec{g}_1$, g_0 are

$$\vec{g}_1 = +\frac{\hbar}{i}\vec{g}$$

$$g_0 = (\frac{\hbar}{2i} + \frac{1}{4}\gamma)div\ \vec{g}$$

The transformations $\mathbb{N}(\kappa, \gamma, r(S), t(S))$ build a group $\mathfrak{N}$.

4.3   Applications of the Nonlinear Tangent Map

We consider the behaviour of $\mathbb{Q}^{(D)}(S_0(R^3))$ under $\mathfrak{N}$. For $D = 0$, i.e. for $\mathbb{Q}^{(0)}(S_0(R^3))$, we have (see (3))

$$\mathsf{Q}(f) = f$$

$$\mathsf{P}^{(0)}(X) = \frac{\hbar}{i}\vec{g} \cdot \nabla - i\frac{\hbar}{2}div\ \vec{g}.$$

Hence with $\gamma = 4D$

$$\mathbb{Q}^{(D)}(S_0(R^3)) = \mathbb{Q}_{\mathbb{N}}^{(0)}(S_0(R^3))$$

holds. Representations of $\mathbb{Q}^{(D)}(S_0(R^3))$ with different D which are inequivalent under linear unitary transformations are 'equivalent' with respect to certain non–unitary ones. It is interesting to apply the $\mathbb{N}$-tangent map to the linear Schrödinger equation. For $\mathbb{N}$ we find that

$$\mathbb{D}_S^0 \circ \mathbb{N}\,[\Psi] = 0, \qquad \mathbb{N} \in \mathfrak{N}$$

leads to a subfamily of generalized NLSE which contain (after gauge generalisation) the DG family. After gauge generalisation a "general" DG family is constructed.

If one applies $\mathbb{N}$ to an ordered polynomial generated from $\mathsf{P}^{(D)}(X)$, $\mathsf{Q}(f)$ one gets a nonlinear quantisation of all observables of polynomial type. The partial Lie algebra structure of these nonlinear operators is known.

5   Summary and Outlook

We started with a geometrical framework to quantise the kinematic of a system living on a topologically nontrivial manifold (Quantum Borel Kinematic). We showed how the quantisation depends on the topology through topological quantum numbers. Since inhomogeneous diffeomorphisms are used as models for the kinematic, a new quantum number of non–topological origin appears. If a time dependence is introduced to the quantized kinematic through conservation of probability, this quantum number is the root of nonlinear Schrödinger equations (DG equations) for pure states. Properties of stationary solutions of the DG equations lead to the introduction of nonlinear gauge transformations which transform a system in a physically equivalent one. Applied to the linear

Schrödinger equation, nonlinear gauge transformations in $\mathcal{G}$ lead after gauge generalisation to the DG equations. A more direct indication of an intrinsic structure of the quantum Borel kinematic utilises $\mathbb{N}$–tangent maps, $\mathbb{N} \in \mathcal{N}$, which transform linear quantised kinematical operators into nonlinear ones. These $\mathbb{N}$–tangent maps can describe a nonlinear quantisation of polynomial observables; for the Hamiltonian a generalised family of DG equations appears.

Topological effects in quantum mechanics and the topological quantum numbers are a well established field with few applications to real systems, e.g. indistinguishable particles in R , anyons, Bohm–Aharonov situations. Nonlinear quantum mechanics for pure states is an interesting but controversial field. On one hand it seems to be plausible that quantum theory is a linearisation of a more involved theory and that nonlinear evolutions derived from first principles are a key stone for a formulation of such a framework. On the other hand we know that linear structures are deeply rooted in the mathematical and physical formulation of quantum theory. The present formalism does not allow nonlinear operators; only for approximations see [37]. A 'new' formalism is not yet developed. There is no experimental indication for a fundamental nonlinearity. However, deviations from usual quantum mechanics are discussed in connection with quantum mechanical precision experiments and with new possibilities for an experimental design. In this connection topological viewpoints as well as nonlinearities in the evolution are of interest.

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